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STEADY-STATE CRACK GROWTH IN SUPERCRITICALLY TRANSFORMING MATERIALS

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Abstract—The steady-state crack growth in an elastic solid which contains particles that undergo irreversible dilatant transformation is analyzed. The transformation is assumed to be uniform and to be triggered when the mean stress attains a critical value. The transformed zone size and the toughening are obtained for the full range of a transformation intensity parameter.

INTRODUCTION

It is well established experimentally (Evans and Heuer, 1980; Evans and Cannon, 1986) that ceramics containing particles that undergo martensitic-type transformation experience fracture toughness increase. Toughening may be measured by the ratio K/K_{tip} of the far-field to the crack tip stress intensity factors. The conditions for the initiation of the transformation are not fully established, but one that has received the most attention is that dilatational transformation occurs when the mean stress reaches some critical value. (Another condition which is mathematically less tractable but may be more realistic is that the transformation occurs when a critical value of the maximum shear stress is attained.)

There have been a number of theoretical studies (e.g. see McMeeking and Evans (1982)) based on the mean stress criterion. Detailed results for the toughening in steady-state crack growth have been calculated (Budiansky *et al.*, 1983) over a wide range of values of an intensity-of-transformation parameter ω to be defined later. In a related study Rose (1986) showed that, for the case of so-called supercritically transforming particles, a crack in the composite can "lock up", with $K/K_{tip} \to \infty$ at a finite value of ω. In this paper we provide results for the toughening and transformation zone size for the full range of values of the intensity parameter ω up to lock-up.

SOME BASIC RELATIONSHIPS

Consider a semi-infinite crack in an elastic solid which contains particles that undergo an irreversible, stress-induced dilatant transformation θ_p^T. The particles are assumed to transform supercritically when the mean stress σ_m attains some critical value σ_m^c. That is, if c is the volume fraction of the particles the dilatant transformation strain $\theta^T = c\theta_p^T$ occurs *discontinuously* in the composite when

$$\sigma_m \equiv \tfrac{1}{3}\sigma_{kk} = \sigma_m^c. \tag{1}$$

As the crack advances under steady-state conditions, transformation occurs along a curve C ahead of the crack tip (see Fig. 1), and in the wake region A behind C no further transformation occurs. The growth of the crack is assumed to occur at a constant critical value of the crack-tip stress intensity factor K_{tip} under a constant applied far-field stress intensity factor K. The stress field remote from the crack tip is given by

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Fig. 1. Transformed zone in steady-state crack growth.

$$\sigma_{ij} = \frac{K}{\sqrt{(2\pi R)}} f_{ij}(\phi) \quad \text{as} \quad R \to \infty \tag{2}$$

where $z = x_1 + ix_2 = R\,e^{i\phi}$ denotes a material point, and the stress field in the neighborhood of the crack tip has the similar form

$$\sigma_{ij} = \frac{K_{\text{tip}}}{\sqrt{(2\pi R)}} f_{ij}(\phi) \quad \text{as} \quad R \to 0. \tag{3}$$

Let $z_0 = R_0\,e^{i\phi_0}$, $\phi_0 > 0$ denote the center of a circular spot of material of area dA that undergoes dilatant transformation θ^T. Under plane strain conditions, the mean stress σ_m at z resulting from a pair of such spots located at z_0 and $\bar{z}_0$, the conjugate of z_0, is (Hutchinson, 1974)

$$\sigma_m = \frac{(1+\nu)E\theta^T\,dA}{18\pi(1-\nu)} \operatorname{Re}\left\{\frac{1}{\sqrt{(zz_0)}(\sqrt{z}+\sqrt{z_0})^2} + \frac{1}{\sqrt{(z\bar{z}_0)}(\sqrt{z}+\sqrt{\bar{z}_0})^2}\right\} \tag{4}$$

where Re denotes the real part. The stress intensity factor induced by the pair of spots is

$$\Delta K_{\text{tip}} \approx \frac{E\theta^T\,dA}{3(1-\nu)\sqrt{(2\pi)}} R_0^{-3/2} \cos \tfrac{3}{2}\phi_0$$

and so the tip stress intensity factor due to the far-field, eqn (2), plus the changes imposed by the transformation in A (Fig. 1) is

$$K_{\text{tip}} = K + \frac{E\theta^T}{3(1-\nu)\sqrt{(2\pi)}} \iint_A R_0^{-3/2} \cos \frac{3\phi_0}{2}\,dA. \tag{5}$$

APPROXIMATE TOUGHENING FORMULA

If the effects of the transformation on the location of the front boundary C of the transformation zone are neglected, i.e. only the far-field stresses, eqn (2), are taken into account in the transformation criterion (1), the shape of C is given by

$$R(\phi) \approx \frac{8}{3\sqrt{3}} H \cos^2 \frac{\phi}{2} \quad (0 \leqslant \phi \leqslant \pi/3) \tag{6}$$

where H is the half-height of the wake, given by

$$H \approx \frac{\sqrt{3}(1+\nu)^2}{12\pi}\left(\frac{K}{\sigma_m^c}\right)^2. \tag{7}$$

Evaluation of the integral in eqn (5), with C prescribed by eqn (6), gives

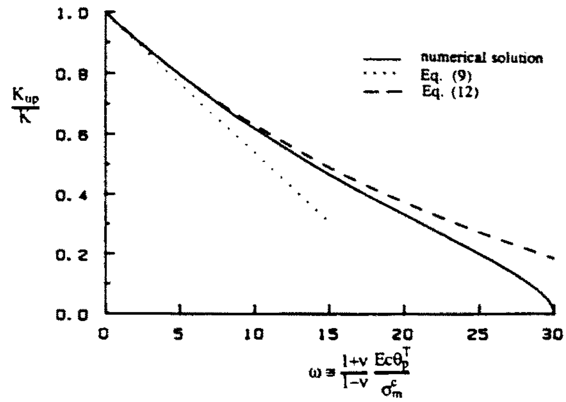

Fig. 2. Dependence of crack-tip stress intensity on intensity of transformation.

$$K_{up} = K - \frac{E\theta^T}{2(1-v)}\left(\frac{H}{\pi\sqrt{3}}\right)^{1/2}. \tag{8}$$

Substitution of estimate (7) for H provides the approximate equation given by Budiansky *et al.*

$$K_{up}/K \approx 1 - \frac{\sqrt{3}}{12\pi}\omega \tag{9}$$

where

$$\omega = \frac{1+v}{1-v}\frac{Ec\theta_p^T}{\sigma_m^c}. \tag{10}$$

This is plotted as the dotted line in Fig. 2. But a much better approximation can be found by exploiting the path-independent J-integral for steady-state cracking (Budiansky *et al.*, 1983) which, for supercritical transformation, provides the *exact* relation

$$\frac{K^2(1-v^2)}{E} = \frac{K_{up}^2(1-v^2)}{E} + 2H\sigma_m^c\theta^T. \tag{11}$$

It may be anticipated that K_{up} depends strongly on H but less so on the detailed shape of C. Accordingly, eliminating H between eqns (8) and (11) gives

$$K_{up}/K = \frac{1 - \frac{\sqrt{3}}{24\pi}\omega}{1 + \frac{\sqrt{3}}{24\pi}\omega} \tag{12}$$

plotted as the dashed curve in Fig. 2. This compares well with numerical results which were found only up to $\omega = 20$ by Budiansky *et al.*, as shown by the solid curve, but does not provide a reliable result for lock-up, when $K_{up}/K = 0$. We will describe next our numerical solution for the rest of the solid curve, up to and including lock-up.

NUMERICAL SOLUTION

The steady-state problem requires that eqn (1) be satisfied as the boundary of the transformation zone is approached from the outside. The mean stress at any point outside the transformed zone consists of the sum of the contributions from eqns (2) and (4), the

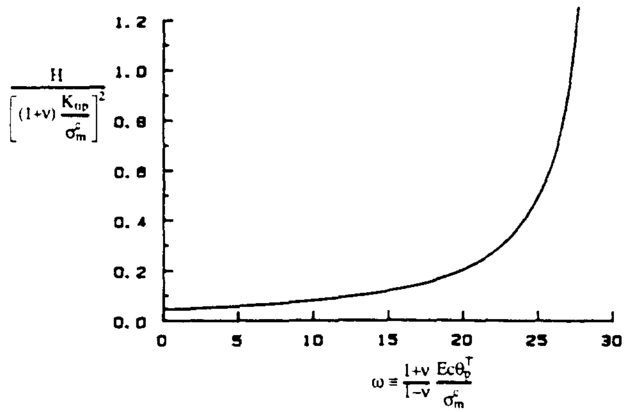

Fig. 3. Dependence of size of transformation zone on intensity of transformation.

latter being integrated over the entire transformed zone A (Fig. 1). Integration with respect to the horizontal coordinate, with use of the substitution

$$r = \frac{9\pi}{2(1+\nu)^2}\left(\frac{\sigma_m^c}{K}\right)^2 R \tag{13}$$

shows that the shape $r(\phi)$ of boundary C satisfies the integral equation (Budiansky *et al.*, 1983)

$$\frac{1}{\sqrt{r}}\cos(\phi/2) + \frac{2\omega}{9\pi}\int_0^{\phi_m} F(\phi,\tilde{\phi})\,\mathrm{d}\tilde{\phi} = 1, \quad 0 \leqslant \phi \leqslant \phi_m \tag{14}$$

where

$$F(\phi,\tilde{\phi}) = \tfrac{1}{2}[g(\phi,\tilde{\phi})+g(\phi,-\tilde{\phi})]\frac{\mathrm{d}}{\mathrm{d}\tilde{\phi}}(\tilde{r}\sin\tilde{\phi})$$

$$g(\phi,\tilde{\phi}) = -\frac{\sqrt{(\tilde{r}/r)}\cos[(\phi+\tilde{\phi})/2]+\cos\phi}{r+\tilde{r}+2\sqrt{(r\tilde{r})}\cos[(\phi-\tilde{\phi})/2]}.$$

Here the polar angle ϕ_m to the top of C must satisfy $(\partial/\partial\phi)(r\sin\phi) = 0$ at ϕ_m. The solution of the integral equation, eqn (14), for $r(\phi)$ and ϕ_m was obtained by writing

$$r(\phi)\sin\phi = \sum_1^N a_n \sin[(n-1/2)\pi\phi/\phi_m], \quad 0 \leqslant \phi \leqslant \phi_m. \tag{15}$$

We substituted series (15) into the integral equation, multiplied the equation by $\sin[(m-1/2)\pi\phi/\phi_0]$ for $m = 1,\ldots,N$ and integrated with respect to ϕ from 0 to ϕ_m to obtain N equations for the $N+1$ unknowns $a_1,\ldots,a_N$ and ϕ_m. The $(N+1)$th equation was obtained by setting $\phi = \phi_m$ in eqn (14). The $N+1$ non-linear equations were then solved by the Newton–Raphson iteration method. (The solution scheme used here is different from that used by Budiansky *et al.*) The crack tip intensity K_{tip} is then computed from the equation

$$K_{tip} = K\left[1 - \frac{2\omega}{9\pi}\int_0^{\phi_m}\frac{1}{\sqrt{r}}\cos\left(\frac{\phi}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\phi}(r\sin\phi)\,\mathrm{d}\phi\right] \tag{16}$$

that follows from eqn (6), again via integration over A in the horizontal direction. The numerical results for K_{tip}/K are exhibited for the full range of values of the intensity-of-transformation parameter ω in Fig. 2. Further, with the wake half-height given by

$$H = R(\phi_m)\sin\phi_m \tag{17}$$

the resulting values of $H/[(1+\nu)K_{tip}/\sigma_m^c]^2$ are shown in Fig. 3.

The value of the transformation intensity at which lock-up occurs, that is, for which $K_{tip}/K = 0$, was obtained by setting the coefficient of K in eqn (16) equal to zero and treating ω as an additional unknown. This provided $N+2$ equations for the $N+2$ unknowns $a_1, \ldots, a_N$, ϕ_m and ω, also solved by the Newton–Raphson method. The result for the critical lock-up parameter is $\omega = \omega_c = 29.99$, which agrees with the value obtained by Rose (1986).

The value of the angle ϕ_m at the top of the transformation zone boundary C was found to vary from $\phi_m = 60^\circ$ at $\omega = 0$ to $\phi_m \approx 97^\circ$ at lock-up, in agreement with Rose. All of the results for K_{tip}/K were obtained to better than three-figure accuracy with $N = 2$.

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